

Geodesic Structure of the Schwarzschild Black Hole in Rainbow Gravity

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In this paper we study the geodesic structure of the Schwarzschild black hole in rainbow gravity analyzing the behavior of null and time-like geodesic. We find that the structure of the geodesics essentially does not change when the semi-classical effects are included. However, we can distinguish different scenarios if we take into account the effects of rainbow gravity. Depending on the type of rainbow functions under consideration, inertial and external observers see very different situations in radial and non radial motion of a test particles.

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I. INTRODUCTION

Nowadays one of the most challenging tasks in Theoretical Physics is to combine quantum theory and general relativity. Several lines of research have been developed but none of them is completely successful in obtaining a complete description of the quantum gravity realm. Meanwhile, some phenomenological approaches have been put on the table. One of them is the modification of the dispersion relation $E^2 - p^2 = m^2$ [1], with a non linear version instead. After all life is not linear at all, so it is very probable that the linear version of the relation linking energy and momenta is just a first approximation to a real non lineal one. This assumption is a base of quantum gravity models, that suggests that it could be desirable to review the Lorentz invariance relations and the very structure of the space time at high energy scales, and not wipe Lorentz symmetry out, but just to modify it to have a non linear version that fits with the usual one at low energies.

On the other hand, some data seem to invite to introduce a minimal length in physical theories. Indeed, there already exist well established theories, such as String Theory or Loop Quantum Gravity, that have some fundamental quantities: the Planck longitude $l_p = \sqrt{\hbar G/c^3}$, the associated time scale $t_p = l_p/c$ and the Planck energy $E_p = \hbar/t_p$. All of them suppose that beyond these thresholds, the physics should change dramatically. However, even discreteness is, in some models, coherent with Lorentz symmetry, since these absolute values of longitude, time or energy are not in total agreement with the Lorentz transformations and this fact is additional motivation to modify the Lorentz boosts.

In fact, among all proposals to deepen our understanding of the nature of spacetime by changing some apparently well settled ideas in Physics, we can find a very interesting one that is to modify the Lorentz boosts through the Double Special relativity (DSR) proposals [2, 3, 4]. These theories are based on a generalization of Lorentz transformations through a more broad point of view of conformal transformations, where there exist two observer independent scales, velocity of light and Planck length. These theories are rather polemical, but are of increasing interest too because they can be useful as effective new tools in gravity theories for example, in Cosmology as an alternative to inflation [5, 6], or in other fields like propagation of light [8], that are related, for instance, to cosmic microwave background radiation.

Finally, it is worthy to study an effect of having a modified dispersion relation (that we expect represents quantum effects), in a strong gravitational field, such as a black hole. In this approach, there are several works about called rainbow gravity, whose history begins more or less with a treatment done in Ref.[9]. In particular, it could be

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interesting in near future, study about the effects of this quantum corrections here proposed, on the Schwarzschild black hole lensing features that were well reviewed on [10] and [11], in the standard case, in order to verify the real existence of the deformations predicted from rainbow gravity formalism.

In this paper we review the structure of geodesics near a Schwarzschild black hole, motivated by the fact that black holes provide gravity conditions to test quantum effects due to the discrete nature of spacetime or the existence of a limit in the energy that a particle can bear. If a testable effect is encountered, we could have a reliable way to examine an underlying hypothesis of a modified dispersion relation.

The paper is organized as follows: In the next section we introduce the rainbow gravity that is a modified version of the Schwarzschild solution of Einstein equation due to generalized dispersion relations above mentioned. In the third section we analyze the geodesic structure, and in the fourth one we discuss conclusions.

II. RAINBOW GRAVITY

Rainbow gravity was proposed in Ref.[9] and it is based on two principles. The first one is the correspondence principle that provides validity of standard general relativity in the limit of low energies relative to the Planck Energy. Then, when $E/E_{pl} \ll 1$, rainbow gravity becomes the standard general relativity. The other one is the Modified Equivalence Principle, that ensures to have a freely falling observer that measures the same laws of physics as in modified special relativity. Therefore, we can construct an energy dependent orthonormal frames locally given by

$$g(E) = \eta^{ab} e_a(E) e_b(E), \quad (1)$$

where the correspondence principle is established through the explicit expression for the orthonormal base

$$\begin{aligned} e_0(E) &= \frac{1}{f(E/E_{pl})} \tilde{e}_0, \\ e_i(E) &= \frac{1}{g(E/E_{pl})} \tilde{e}_i. \end{aligned} \quad (2)$$

Having Minkowski space in the low energy limit $E/E_{pl} \rightarrow 0$ implies the following relation between the arbitrary functions $f(E/E_{pl})$ and $g(E/E_{pl})$

$$\lim_{E/E_{pl} \rightarrow 0} f(E/E_{pl}) = \lim_{E/E_{pl} \rightarrow 0} g(E/E_{pl}) = 1. \quad (3)$$

On the other hand, arbitrary energy dependent function originates in the modification of Lorentz dispersion relation

$$E^2 f(E/E_{pl}) - p^2 h(E/E_{pl}) = m_0^2, \quad (4)$$

that explicitly shows a deformed nature, or doubly special relativity, as a class of theories that implement a changed set of principles to special relativity. From the above equations we see that the metric (1) that describes a flat rainbow space, can be generalized to a curve rainbow space. Similarly as in the flat case, it corresponds to a one-parameter family of metrics given in terms of one-parameter family of orthonormal frame fields (2), whose energy-dependent metric must satisfy a modified version of Einstein Equations given by

$$G_{\mu\nu}(E) = 8\pi G_N(E) T_{\mu\nu}(E) + g_{\mu\nu}(E) \Lambda(E). \quad (5)$$

Here, the Newton constant $G_N(E)$ and the cosmological constant (for asymptotically Anti de Sitter or de Sitter space) $\Lambda(E)$ are allowed to be energy-dependent, and they satisfy the correspondence principle.

In this approach in Ref. [9], it was presented a modified general spherically symmetric solution to equations (5), known as Schwarzschild modified Black Hole, described by the metric

$$ds_{Schw}^2 = -\frac{(1 - \frac{2G(0)M}{\tilde{r}})}{f^2(E/E_{pl})} d\tilde{t}^2 + \frac{1}{(1 - \frac{2G(0)M}{\tilde{r}})g^2(E/E_{pl})} d\tilde{r}^2 + \frac{\tilde{r}^2}{g^2(E/E_{pl})} d\tilde{\Omega}^2, \quad (6)$$

where the quantities $(\tilde{t}, \tilde{r}, \tilde{\Omega})$ are independent energy variables. The metric also depends on the energy of a particle moving in it. That is, two different test particles observe different effective space-time geometries. As a consequence, the present space-time is endowed with a Plank-scale modification that carries some quantum effects. We are interested in studying a behavior of a test particle under the influence of this geometry with quantum corrections. This point will be analyzed in next section.

III. GEODESICS

Using the variational principle [12, 13], the metric (6) is associated with a Lagrangian density \mathcal{L} given by

$$2\mathcal{L} = -\frac{F(r)}{f_E^2}\dot{t}^2 + \frac{1}{g_E^2 F(r)}\dot{r}^2 + \frac{r^2}{g_E^2}\dot{\Omega}^2, \quad (7)$$

where $f_E \equiv f(E)$ and $g_E \equiv g(E)$, whereas $F(r)$ is the usual lapses function of the Schwarzschild spacetime ¹

$$F(r) = 1 - \frac{r_+}{r}, \quad (8)$$

and $\dot{\Omega}^2 = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2$. In this notation dot represents a derivative with respect to proper time, τ (affine parameter along a geodesic). Since the Lagrangian does not depend on (t, ϕ) , the corresponding conjugate momenta are conserved, therefore in the invariant plane $\theta = \pi/2$ we have

$$\Pi_t = -\frac{F(r)}{f_E^2}\dot{t} = -E, \quad (9)$$

and

$$\Pi_\phi = \frac{r^2}{g_E^2}\dot{\phi} = L, \quad (10)$$

where E and L are constants of motion. From the last two equations, and taking $\bar{\tau} = g_E\tau$, $\mathbb{E} = f_E E$ and $\mathbb{L} = g_E L$, the Lagrangian (7) can be written in the following form

$$2\mathcal{L} \equiv -m = -\frac{\mathbb{E}^2}{F(r)} + \frac{\dot{\bar{r}}^2}{F(r)} + \frac{\mathbb{L}^2}{r^2}, \quad (11)$$

where, by normalization, $m = 1$ for massive particles (time-like geodesics) and $m = 0$ for massless particles (null geodesics), and $\dot{\bar{r}} = dr/d\bar{\tau}$. Thus, our equation of motion becomes

$$\dot{\bar{r}}^2 = \mathbb{E}^2 - \mathbb{V}_G(r; m, \mathbb{L}), \quad (12)$$

where $\mathbb{V}_G(r; m, \mathbb{L})$ is the generalized effective potential, which is given by

$$\mathbb{V}_G(r; 0, \mathbb{L}) \equiv \mathbb{V}_N(r; \mathbb{L}) = F(r) \frac{\mathbb{L}^2}{r^2}, \quad (13)$$

for null geodesics, and

$$\mathbb{V}_G(r; 1, \mathbb{L}) \equiv \mathbb{V}_T(r; \mathbb{L}) = F(r) \left(1 + \frac{\mathbb{L}^2}{r^2} \right), \quad (14)$$

for time-like geodesics. In what follows, we use obtained results to discuss two families of functions, say, DSR1 with $f_E = 1$ and $g_E = 1 + \frac{1}{2}l_p E$; and DSR2 with $f_E = g_E (= 1 + \frac{1}{2}l_p E)$ [7].

A. Null Geodesics

Related to the equation of motion for massless particle, we start from Eqs. (12) and (13), and we study independently the radial and non-radial motion.

(a.i).- Radial Null Geodesics

¹ This treatment is valid for the Kottler spacetime if cosmological constant does not depend on the constant of motion E , in which case a complete analytic solution of the geodesic structure of the Schwarzschild anti-de Sitter was done in [13], with the lapses function given by $F(r) = 1 - \frac{r_+}{r} + \frac{r^2}{\ell^2}$, where $\frac{\Lambda}{3} = -\frac{1}{\ell^2}$.

In this case we have $\mathbb{V}_N(r; \mathbb{L}) = 0$ and the radial motion is governed by

$$\dot{r}^2 = \mathbb{E}^2, \quad (15)$$

therefore

$$\Delta\tau = \frac{(\Delta r/E)}{\Gamma_1} = \frac{\Delta\tau_{Schw}}{\Gamma_1}, \quad (16)$$

where $\Gamma_1 = g_E f_E$. We see that the radial motion of massless particle shows the same behavior as standard Schwarzschild geometry, with the only difference that its proper time is rescaled by a factor Γ_1 .

Furthermore, from Eqs. (9) and (15), we find an expression for the coordinate time, t ,

$$\frac{dt}{dr} = \frac{\Gamma_2}{F(r)}, \quad (17)$$

in which case we obtain

$$\Delta t = \Gamma_2 \Delta t_{Schw} = \Gamma_2 \left[(r_i - r) + r_+ \log \left(\frac{\frac{r}{r_+} - 1}{\frac{r_i}{r_+} - 1} \right) \right], \quad (18)$$

where $\Gamma_2 = \frac{g_E^3}{f_E}$. This situation is showed in Fig. 1 in case of DSR1 and DSR2, in the limit $E \rightarrow E_p (= l_p^{-1})$, together with the semi-classical limit (Schwarzschild case).

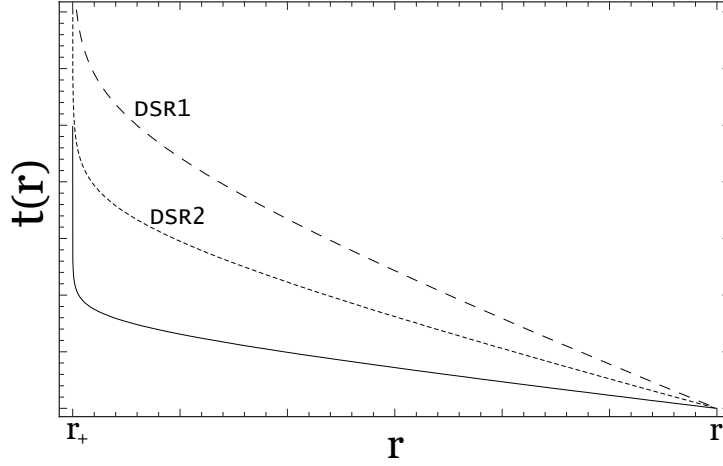


FIG. 1: This plot shows the coordinate time t versus the radial distance from the black hole, r , for massless particles falling in radial motion. The solid curve represents the semi-classical limit (Schwarzschild case). The dashed curves represents the modifications DSR1 and DSR2 in the limit $E \rightarrow E_p (= l_p^{-1})$.

(a.ii).- Non-Radial Null Geodesics

Returning to the equation of motion (12-13), it is convenient to rewrite it in terms of the new variable $u = 1/r$, and using (10) it can be put in the form analog to the one shown in Ref.[12]

$$\left(-\frac{du}{d\phi} \right)^2 = r_+ u^3 - u^2 + \frac{1}{\bar{b}^2}. \quad (19)$$

Here $\bar{b} = \sqrt{\Gamma_3} b$ denotes the generalized impact parameter for the orbital motion of the massless particles, and $\Gamma_3 = \Gamma_2/\Gamma_1 = g_E^2/f_E^2$. Note that, from the equation (19), the non-radial motion is identical to the one corresponding to the Schwarzschild black hole when the DSR2 deformation is considered.

B. Time-Like Geodesics

The motion of a massive particle is described by Eqs. (12-14)

$$\dot{r}^2 = \mathbb{E}^2 - F(r) \left(1 + \frac{\mathbb{L}^2}{r^2} \right), \quad (20)$$

(b.i).- Radial Time-Like Geodesics

In this case, the equation of motion (20) can be written as

$$\left(\frac{dr}{d\bar{\tau}} \right)^2 = \frac{r_+}{r} - (1 - \mathbb{E}^2). \quad (21)$$

Making the usual substitution

$$R = \frac{r}{r_i} = \cos^2 \frac{\eta}{2}, \quad (22)$$

where ($R_+ = \frac{r_+}{r_i} < u \leq 1$), we obtain for the proper time and coordinate time respectively

$$\tau = \frac{r_i}{g_E} \Theta_{l1}, \quad (23)$$

$$t = r_i \frac{f_E \mathbb{E}}{g_E} \Theta_{l2}, \quad (24)$$

where we introduce the functions

$$\Theta_{l1} = \frac{1}{\sqrt{R_+}} \left[\arccos \sqrt{R} + \sqrt{R - R^2} \right], \quad (25)$$

and

$$\Theta_{l2} = \frac{1}{\sqrt{R_+}} \left[\sqrt{R - R^2} + (2R_+ + 1) \arccos \sqrt{R} + \frac{2R_+^2}{\sqrt{R_+ - R_+^2}} \operatorname{arctanh} \sqrt{\frac{R_+(1 - R)}{R(1 - R_+)}} \right]. \quad (26)$$

The returning point $r_i = \frac{r_+}{1 - \mathbb{E}^2}$ was chosen as the starting point, $r_0 = r_i$. To have positive r_i , we have to impose $\mathbb{E}^2 < 1$, and we shall assume that $E \lesssim 1$ (Schwarzschild case $E < 1$). This means that energies are low $E \ll E_p$ and the corrections from the rainbow gravity are negligible.

Consider now a sector out of the capture zone, i.e, where $\mathbb{E}^2 > 1$. In this case, the change of radial variable suggested by (21) is

$$R = \frac{r}{r_e} = \sinh^2 \frac{\xi}{2}, \quad (27)$$

where $r_e = \frac{r_+}{\mathbb{E}^2 - 1}$ is a distance-energy parameter of a falling particle. Therefore, for the particle falling from the distance r_0 ($R_0 = r_0/r_e$), we have

$$\tau = \frac{r_e}{g_E} [\Theta_{g1}(R) - \Theta_{g1}(R_0)], \quad (28)$$

and

$$t = r_e \frac{f_E \mathbb{E}}{g_E} [\Theta_{g2}(R) - \Theta_{g2}(R_0)], \quad (29)$$

where

$$\Theta_{g1}(R) = \frac{1}{\sqrt{R_+}} \left[\operatorname{arcsinh} \sqrt{R} - \sqrt{R + R^2} \right], \quad (30)$$

and

$$\Theta_{g^2}(R) = \frac{1}{\sqrt{R_+}} \left[(2R_+ + 1) \operatorname{arcsinh} \sqrt{R} - \sqrt{R + R^2} + \frac{4R_+^2}{\sqrt{R_+ + R_+^2}} \operatorname{arctanh} \sqrt{\frac{R_+(1-R)}{R(1-R_+)}} \right]. \quad (31)$$

In Fig. 2 we show the proper and coordinate times for radial massive particle freely falling into the black hole. Remarkable feature is that, in comparison with the null radial motion, the DSR1 and DSR2 modifications for time-like motion coordinate times, both merged in one curve.

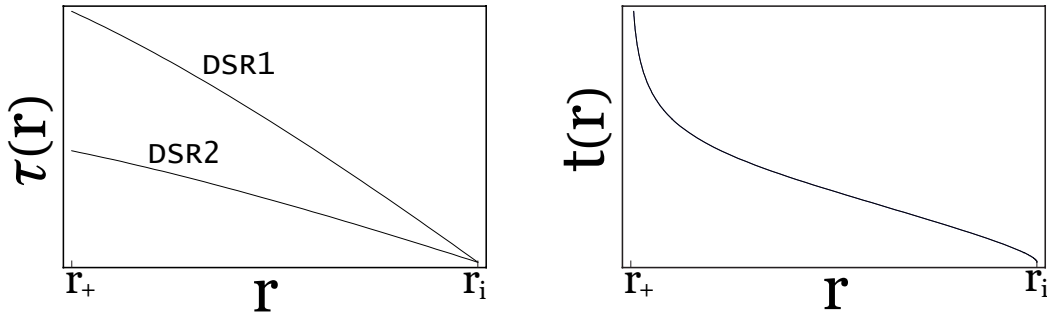


FIG. 2: This plot shows a behavior of a massive particle radially falling into the black hole in the region out of the capture zone, i.e. when $E > 1$. Left panel shows the proper time τ as a function of a distance r , and it has larger value in the DSR1 than DSR2 scenario in the limit $E \rightarrow E_p (= l_p^{-1})$. Right panel shows coordinate time t versus radial coordinate r . In the limit $E \rightarrow E_p$, both cases of DSR1 and DSR2 scenarios, converge to the same function. This means that an outside observer can not see differences between the DSR1 and DSR2 modifications.

(b.ii).- Non-Radial Time-Like Geodesics

Using (14) and defining $r = 1/u$, the motion of a non-radial massive particle is governed by equation

$$\left(\frac{du}{d\phi} \right)^2 = r_+ u^3 - u^2 + \frac{r_+}{\mathbb{L}^2} u - \frac{1 - \mathbb{E}^2}{\mathbb{L}^2}. \quad (32)$$

Again, if we consider lower energies ($\mathbb{E} \sim E$ and $\mathbb{L} \sim L$), we obtain the corrections from rainbow gravity are negligible. This means that bounded orbits for massive particles are not affected by the effects of rainbow gravity.

IV. DISCUSSION AND OUTLOOK

We study the geodesic structure of Schwarzschild black holes in rainbow gravity, analyzing the behavior of null and time-like geodesics for DSR1 and DSR2 theories. We found that the structure of geodesics does not change when semi-classical effects are taken into account. The case of radial null-geodesics shows that the effects of a space-time endowed with a Planck-scale modification, and therefore including the quantum effects, are of kinematic origin and the only correction is an adding an another proper time contraction equation (16). For the coordinate time Eq.(18), in both theories we found larger values of times than in standard Schwarzschild case. Our results in that case are summarized in Fig. 1, for both DSR1 and DSR2. The photons with energy of order of the Planck scale exhibit a modification in the Doppler effect, as seen from outside. For a non radial-null geodesic, the only modification comes from the impact parameter $\bar{b} = \sqrt{g_E^2/f_E^2} b_{schw}$. In the case of a massive particle we found, for radial geodesics, that the only modification is in a changed returning point, whereas for a non-radial geodesic, the effects depend on the relation between $g(E)$ and $f(E)$ under consideration.

Therefore, we conclude that different test particles (with different energies) do not see different spacetimes. Based on our results, different test particles have different effective descriptions, that means that there are some changes in their kinematics properties around the same rainbow Schwarzschild black hole when the quantum effects have been taken in to account.

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